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The Electromagnetic Energy and Momentum of Finite Charged Bodies Moving at Constant Velocity

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Abstract

It is shown that a purely electromagnetic, divergence-free tensor S^{ij}, can be defined for any electrically charged body which is held in equilibrium by some cohesive force and moving at some constant velocity. This tensor appears to represent the electromagnetic energy-momentum of the body; the integral $(1/c) \int S^{ij} dS_j$ (dS_j is the differential element of any spacelike hypersurface) is $cM_0\mu^i$ the electromagnetic four-momentum of the system (M_0 is the electromagnetic rest mass of the system, U^i is the four-velocity). The divergence-free property of S^{ij} depends only on Maxwell's equation and the condition of uniform motion.

It is suggested that whatever the nature of the cohesive forces within such a system the total stress-energy tensor will, in effect, break up into two parts which are separately divergence-free: the purely electromagnetic tensor, S^{ij}, and a tensor representing the energy-momentum of the cohesive forces. Just as it makes sense to speak of the electromagnetic mass of a system at rest without regard to the cohesive forces, it makes sense to talk about the electromagnetic momentum of the system, when it is moving at constant velocity, without reference to the cohesive forces.

1. *Introduction*

The problem of how the electromagnetic energy of a body (apart from the non-electromagnetic energy of the body), in equilibrium, transforms in special relativity has an extensive literature and is treated in most modern textbooks. Nevertheless, there still appears to be some unresolved issues. The electromagnetic rest energy of a body, M_0c^2 (the subscript zero refers to the rest frame of reference), is conventionally defined as the integral over all space of the squared magnitude of the electric field multiplied by $(8\pi)^{-1}$. Usually one demands that the electromagnetic scalar potential, ϕ_0, vanish at infinity. The rest mass energy can then be represented as the integral of $\sigma_0\phi_0/2$ over the volume of the body; σ_0 is the rest frame charge density. We would expect that when the body is examined in a reference frame (the observer's frame) in which

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the body is viewed to be moving at a constant velocity, **V** (time varying velocities are not considered in this paper), the description of its electromagnetic energy and momentum could be made in terms of a single four-vector P^i where

$$P^i = cM_0\mu^i \tag{1.1}$$

(We adopt the convention that Latin indices take the values 0 to 3 ($X^0 = ct$) and Greek indices take the value 1 to 3; repeated indices are summed over their respective ranges.) Here μ^i is the four-velocity $\gamma(1, \mathbf{V}/c)$ and γ is $(1 - V^2/c^2)^{-1/2}$. Unfortunately it is not a simple matter to separate the electromagnetic energy-momentum from the non-electromagnetic (cohesive) energy-momentum of a body.

One usually defines the energy-momentum four-vector of a body, P_t^i, from the total energy-momentum tensor, I^{ij}, of that body. If I^{ij} vanishes properly at infinity then Gauss's theorem in four dimensions combined with the equations of motion of the body, $I^{ij}{}_{,j} = 0$ (a comma preceding j means differentiation with respect to the coordinate j) show that

$$P_t^i \equiv \frac{1}{c}\int_s I^{ij} dS_j \tag{1.2}$$

is not only conserved but independent of the spacelike hypersurface (Moller, 1952) dS_j (normally the hypersurface $(1, 0)\, d^3X$ is chosen for mathematical convenience; d^3X is the three-dimensional volume element in the observer's frame of reference).

For a charged body I^{ij} is equal to the Maxwell stress-energy tensor plus contributions from non-electromagnetic forces (negative pressure for example). The contribution to I^{ij} from the non-electromagnetic forces will be denoted by t^{ij}.

In order to define the electromagnetic energy-momentum without reference to other forces researchers have worked with the Maxwell stress-energy tensor alone. Abraham's (Jackson, 1962) definition of the electromagnetic energy-momentum,

$$P^i = \frac{1}{c}\int T^{i0} d^3X \tag{1.3}$$

is not a four-vector and so is deemed not to be satisfactory, Rohrlich (1960, 1970) has succeeded in obtaining a four-vector having the desired form of equation (1.1) by specifying a particular hypersurface to integrate on. His definition is

$$P^i = \frac{-1}{c}\int T^{ij} dS_j \tag{1.4}$$

where

$$dS_j = \gamma^2(-1, \mathbf{V}/c)\, d^3X \tag{1.5}$$

It would seem that one should also be able to obtain an energy-momentum four-vector for the electromagnetic portion of a body without reference to a specific hypersurface. In the first part of this paper, we will show that there exists a purely electromagnetic tensor S^{ij}, for any charged body in equilibrium, which is divergence-free ($S^{ij}{}_{,j} = 0$). By Gauss's theorem we can then define a hypersurface invariant electromagnetic energy-momentum vector,

$$P^i = \frac{1}{c}\int S^{ij} dS_j \tag{1.6}$$

which has the form of equation (1.1). S^{ij} will be defined in terms of the electromagnetic potentials of the system and thus without reference to the cohesive forces. The divergence-free property depends only upon the field equations and the uniform motion.

The question then arises as to how S^{ij} is related to I^{ij}. In the Appendix, it is shown that if t^{ij} is the stress-energy tensor of a perfect fluid containing a negative pressure then, using the equations of motion,

$$I^{ij} = \epsilon\mu^i\mu^j + S^{ij} + K^{ij} \tag{1.7}$$

where ϵ is the rest energy density of the fluid and K^{ij} is a divergence-free tensor which vanishes when integrated over a spacelike hypersurface. The tensor K^{ij} does not contribute to the energy-momentum of the system; the fluid mass and the electromagnetic mass can thus be viewed as contributing separately to the total energy-momentum of the system. The t^{ij} for a fluid has the form:

$$t^{ij} = (\epsilon + P)\mu^i\mu^j + P\delta^{ij} \tag{1.8}$$

where P is the pressure. (A system where the cohesive forces are represented by a stress energy tensor of the form of equation (1.8) corresponds to Poincare's fluid model of the electron (Moller, 1952, pp. 192–194). The analysis of the Appendix demonstrates that any rest frame system with an arbitrary spherical charge distribution—not just a rest frame sphere whose charge is uniformly distributed on the surface—can serve as a classical electron model if the repulsive electrostatic forces can be balanced by a negative pressure; this is one result of equation (1.7). The interesting thing about this fluid model for the cohesive forces is that t^{ij} does not vanish if $\epsilon \to 0$ in equation (1.8). Thus, we are left with an electron whose total four-momentum is just given by the electromagnetic four-momentum [see equation (1.7), $\epsilon \to 0$] since the pressure does not contribute to the total momentum of the particle).

If one were able to write tensors t^{ij} for forces other than pressure, the results would probably be the same. I^{ij} would have the same form as equation (1.7) where the total rest energy density, ϵ', of the non-electromagnetic forces would replace ϵ for the fluid; K^{ij} would probably not contribute to the total energy-momentum of the system. Since $\epsilon'\mu^i\mu^j$ and S^{ij} would be separately divergence-free, we have a rationale for speaking of the electromagnetic four-momentum of any rigid, uniformly moving object without reference to the cohesive forces.

2. *The Energy-Momentum Four-Vector*

In this section we consider an electrically charged body, in internal static equilibrium, moving uniformly with a velocity $\mathbf{V}$. The charge on the body is described by the charge density, σ. We will show that a divergence-free tensor, S^{ij}, expressed in terms of the electromagnetic potentials, $(\phi, \mathbf{A})$, and the current vector, J^i, can be defined. Defining the electromagnetic energy-momentum vector of the body by equation (1.6) we will obtain the usual form of an energy-momentum four-vector, as expressed by equation (1.1). We now define S^{ij}. The electromagnetic potentials for a body, every point of which is moving at the same constant velocity, are (Panofsky & Phillips, 1962; Landau & Lifshitz, 1962)

$$A^i = \phi(1, \mathbf{V}/c) = \gamma^{-1}\phi\mu^i \tag{2.1}$$

where

$$\phi(X, t) = \int \frac{\sigma(\mathbf{X}, t)d^3X}{R} \tag{2.2}$$

and

$$R^2 = [(\mathbf{X} - \mathbf{X}) \cdot \mathbf{V}/|\mathbf{V}|]^2 + \gamma^{-2}\left[(\mathbf{X} - \mathbf{X}) \times \frac{\mathbf{V}}{|\mathbf{V}|}\right]^2 \tag{2.3}$$

($\mathbf{X}$ denotes the spatial coordinate vector in the observer's frame.) The current vector is defined by the equation

$$J^i = \sigma(c, \mathbf{V}) = \sigma c\mu^i/\gamma \tag{2.4}$$

S^{ij} is defined as

$$S^{ij} = (4C)^{-1}(A^iJ^j + A^jJ^i) \tag{2.5}$$

Substituting equations (2.1) and (2.4) into (2.5) we obtain

$$S^{ij} = (2\gamma^2)^{-1}\sigma\phi\mu^i\mu^j \tag{2.6}$$

If we express both σ and ϕ in terms of their respective rest system values, σ_0 and ϕ_0, S^{ij} takes a very suggestive form. These transformations to the rest frame are given by

$$c\sigma_0 = c\sigma\gamma^{-1}, \qquad \phi_0 = \phi\gamma^{-1} \tag{2.7}$$

and express the transformations of the zeroth four-vector components of the vectors J^i and A^i, respectively (ϕ_0 has the form of the usual Coulomb potential). With the use of equation (2.7), (2.6) takes the form

$$S^{ij} = (\phi_0\sigma_0/2)\mu^i\mu^j \tag{2.8}$$

Since the electromagnetic mass in the rest frame, M_0, is defined by the equation

$$M_0 = (2c^2)^{-1} \int \sigma_0\phi_0 d^3X_0 \tag{2.9}$$

$\sigma_0\phi_0(c^2 2)^{-1}$ represents the electromagnetic rest mass density. Thus equation (2.8) has the form of the energy-momentum tensor of a body of rest mass M_0 moving at constant velocity.

It will now be shown that S^{ij} is divergence-free. Because the system is in equilibrium, any quantity, Q, associated with a material particle remains constant in time. This is expressed by the equation

$$\frac{dQ}{dt} = Q_{,\alpha}V^{\alpha} + cQ_{,0} = (\gamma/c)Q_{,i}\mu^i = 0 \tag{2.10}$$

Taking the divergence of equation (2.6), substituting equation (2.10) into the resulting expression [where σ and ϕ replace Q in equation (2.10)], while noting that **V** is constant, it is easily seen that $S^{ij}{}_{,i}$ is just zero. Substituting equation (2.8) into equation (1.6) and taking $dS_j = (1, 0)d^3X$ we find that

$$p^i = (\mu^i/c^2) \int \sigma_0\phi_0\mu^0 d^3X = M_0 c\mu^i \tag{2.11}$$

where we have used the fact that $\mu^0 d^3X = \gamma d^3X = d^3X_0$. (Note that if p^i is denoted by $M(c, \mathbf{V})$ then from equations (2.6) and (1.6), M is expressed in the observer's coordinates, by

$$M = \tfrac{1}{2}c^{-2} \int \sigma\phi\, d^3X \tag{2.12}$$

where ϕ is given by equation (2.2).) Equation (2.11) is the usual four-vector expression for the energy-momentum of a moving body.

3. *Concluding Remarks*

We have shown that for a body held in equilibrium by some unspecified cohesive force, there exists a purely electromagnetic tensor S^{ij} which can be used to define the electromagnetic energy-momentum vector without reference to a particular hypersurface if the body is moving uniformly. A relationship between S^{ij} and the total energy-momentum tensor of the system was suggested. This relationship would allow the cohesive and electromagnetic constituents of the four-momentum of the system to be treated separately. Whether a similar tensor can be utilized in studying accelerated motion or motion within the framework of general relativity may be worthy of consideration.

Appendix A

It will be shown that the total energy-momentum tensor for a fluid system has the form

$$t^{ij} + T^{ij} = \epsilon\mu^i\mu^j + S^{ij} + L^{ij\lambda}_{,\lambda} \tag{A.1}$$

where t^{ij} is given by equation (1.8), S^{ij} by equation (2.6) and T^{ij} is the Maxwell stress-energy tensor given by

$$T^{ij} = (4\pi)^{-1}(F^{is}F^j{}_s - \tfrac{1}{4}\delta^{ij}F^{sp}F_{sp}) \tag{A.2}$$

In equation (A.2), F^{ij} is defined by

$$F^{ij} \equiv A_{j,i} - A_{i,j} \tag{A.3}$$

and δ^{ij} by

$$\delta^{ij} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.4}$$

From equation (A.1) one can see that $L^{ij\lambda}_{,\lambda}$ is a symmetric tensor since all the other terms in equation (A.1) are symmetric tensors. It is also clear from equation (A.1) that $L^{ij\lambda}_{,\lambda}$ will not contribute to the total energy-momentum vector if it vanishes sufficiently rapidly at infinity (see equation (1.2)).

We proceed by reducing the left-hand side of equation (A.1) with the aid of the equations of motion, and finding $L^{ij\lambda}_{,\lambda}$ explicitly in terms of P, V^α and ϕ, in a particular but otherwise arbitrary frame of reference. T^{ij} is first expressed in terms of V^α and ϕ. Substituting equations (2.1) and (2.10) ($Q = \phi$) into equation (A.2) while noting that V^α is a constant, yields

$$8\pi T^{00} = (\phi_{,\alpha})^2(V^2/c^2 + 1) + (\phi_{,0})^2(-3 + V^2/c^2) \tag{A.5}$$

$$4\pi T^{0\beta} = V^\beta/c[(\phi_{,\alpha})^2 - (\phi_{,0})^2] - \phi_{,0}\phi_{,\beta}/\gamma^2 \tag{A.6}$$

$$4\pi T^{\alpha\beta} = V^\alpha V^\beta/c^2[(\phi_{,\lambda})^2 - (\phi_{,0})^2] + \phi_{,\alpha}\phi_{,\beta}/\gamma^2 + [\tfrac{1}{2}\delta^{\alpha\beta}/\gamma^2][(\phi_{,\lambda})^2 - (\phi_{,0})^2] \tag{A.7}$$

We now look at the (0, 0)-component of equation (A.1). From equation (1.8) we find that

$$t^{00} + T^{00} = (\epsilon + P)\mu^0\mu^0 - P + T^{00} = \epsilon\mu^0\mu^0 + \frac{V^2}{c^2}\gamma^2 P + T^{00} \tag{A.8}$$

The last two terms on the right-hand side of equation (A.8) are reduced to $S^{00} + L^{00\alpha}_{,\alpha}$ by means of the equations of motion:

$$t^{ij}_{,j} + T^{ij}_{,j} = 0 \tag{A.9}$$

By an argument similar to that which showed S^{ij} to be divergence-free, it can be shown that

$$[(\epsilon + P)\mu^i\mu^j],j = 0 \tag{A.10}$$

Using equation (A.10), equation (A.9) reduces to

$$(P\delta^{ij}),j + T^{ij}_{,j} = 0 \tag{A.11}$$

Using equation (2.10), the (0)-component of equation (A.11) becomes

$$\left[\frac{V^\alpha}{c}(P - T^{00}) + T^{0\alpha}\right]_{,\alpha} = 0 \tag{A.12}$$

With the use of equation (A.12) it is easy to see that

$$B^\beta - T^{0\beta} + \frac{V^\beta}{c} T^{00} = \frac{V^\beta}{c} P \tag{A.13}$$

where

$$B^\beta = \left[X^\beta\left(\frac{V^\alpha}{c}P - \frac{V^\alpha}{c}T^{00} + T^{0\alpha}\right)\right]_{,\alpha} \tag{A.14}$$

and X^β refers to the β-component of the spatial coordinates. Multiplying equation (A.13) by $(V^\beta/c)\gamma^2$ and substituting the result in equation (A.8) we find that

$$t^{00} + T^{00} = \epsilon\mu^0\mu^0 + \gamma^2\left(T^{00} - \frac{V^\alpha}{c}T^{0\alpha}\right) + \frac{V^\beta}{c}B^\beta\gamma^2 \tag{A.15}$$

We now reduce the second term on the right-hand side of equation (A.15). From equations (A.5) and (A.6) it is seen that

$$\gamma^2\left(T^{00} - \frac{V^\alpha}{c}T^{0\alpha}\right) = C(\phi/8\pi(\phi_{,\alpha\alpha} - \phi_{,00})) \tag{A.16}$$

where

$$C = \frac{1}{8\pi}\left[(\phi_{,\alpha})(\phi_{,0}) + (\phi_{,0})\phi V^\alpha/c\right]_{,\alpha} \tag{A.17}$$

The second term on the right-hand side of (A.16) can be shown to be S^{00} by noting that our potentials satisfy Maxwell's equations in the Lorentz gauge. The Lorentz condition is expressed by the equation

$$A^i_{,i} = 0 \tag{A.18}$$

Substituting equation (2.1) into equation (A.18) while taking account of equation (2.10) we find that

$$A^i_{,i} = \frac{1}{c}\frac{d\phi}{dt} = 0 \tag{A.19}$$

This simply means that the scalar potential associated with a material particle does not vary with time. With equation (A.18) substituted in Maxwell's equations, we find that ϕ satisfies the equation

$$\phi_{,\alpha\alpha} - \phi_{,00} = -4\pi\sigma \tag{A.20}$$

Substituting equation (A.20) into equation (A.16) and the result into equation (A.15) we obtain the (0, 0) component of equation (A.1) where

$$L^{00\alpha}_{,\alpha} = C + \frac{V^\beta}{c}\gamma^2 B^\beta \tag{A.21}$$

and C and B^β are given by equations (A.17) and (A.14), respectively.

The $(0, \alpha)$ and (α, β)-components of equation (A.1) can be handled in much the same way as the (0, 0)-component. By means of the spatial part of the equation of motion (see equation (A.11)) we find that

$$t^{0\alpha} + T^{0\alpha} = \epsilon\mu^0\mu^\alpha + \gamma^2\left(T^{\alpha 0} - \frac{V^\lambda}{c}T^{\alpha\lambda}\right) + \gamma^2\frac{V^\lambda}{c}D^{\alpha\lambda} \tag{A.22}$$

and

$$\begin{aligned} t^{\alpha\beta} + T^{\alpha\beta} &= \epsilon\mu^\alpha\mu^\beta + \gamma^2\left(\frac{V^2}{c^2}\frac{V^\beta}{c}T^{\alpha 0} - \frac{V^\beta}{c}\frac{V^\lambda}{c}T^{\alpha\lambda}\right) \\ &\quad + \frac{V^\beta}{c}T^{\alpha 0} + D^{\alpha\beta} + \frac{\gamma^2}{c^2}V^\beta V^\lambda D^{\alpha\lambda} \end{aligned} \tag{A.23}$$

where

$$D^{\alpha\beta} = \left[X^\beta\left(P\delta^{\alpha\lambda} + T^{\alpha\lambda} - \frac{V^\lambda}{c}T^{\alpha 0}\right)\right]_{,\lambda} \tag{A.24}$$

By means of equations (A.6)-(A.7) it can be shown that

$$\gamma^2\left(T^{\alpha 0} - \frac{V^\lambda}{c}T^{\alpha\lambda}\right) = S^{0\alpha} + \frac{V^\alpha}{c}C \tag{A.25}$$

and

$$\gamma^2\left(\frac{V^2}{c^2}\frac{V^\beta}{c}T^{\alpha 0} - \frac{V^\beta}{c}\frac{V^\lambda}{c}T^{\alpha\lambda}\right) + \frac{V^\beta}{c}T^{\alpha 0} = S^{\alpha\beta} + \frac{V^\beta V^\alpha}{c^2}C \tag{A.26}$$

where C is defined by equation (A.17). We therefore see, by substituting equation (A.25) into equation (A.22) and equation (A.26) into equation (A.23), that

$$L^{0\alpha\lambda}_{,\lambda} = \gamma^2\frac{V^\lambda}{c}D^{\alpha\lambda} + \frac{V^\alpha}{c}C \tag{A.27}$$

and

$$L^{\alpha\beta\lambda}_{,\lambda} = D^{\alpha\lambda} + \frac{\gamma^2}{c^2}V^\beta V^\lambda D^{\alpha\lambda} + \frac{V^\beta V^\alpha}{c^2}C \tag{A.28}$$

where $D^{\alpha\beta}$ is defined by equation (A.24). If the spatial dependence of $L^{ij\lambda}_{,\lambda}$ is examined it is seen that for large distances from the origin, R, $L^{ij\lambda}_{,\lambda}$ goes approximately as $1/R^3$; $L^{ij\lambda}$ does not, therefore, contribute to the energy-momentum vector of the system. It can also be shown explicitly that $L^{ij\lambda}_{,\lambda}$ is divergence-free. The $L^{ij\lambda}_{,\lambda}$ found in this appendix are just the K^{ij} of equation (1.7), for a fluid system, in the observer's system of coordinates.

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